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COMPARING THE RELATIVE VOLUME WITH A REVOLUTION MANIFOLD AS A MODEL

BY

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ABSTRACT

Given a pair (P, M), where M is an n-dimensional connected compact Riemannian manifold and P is a connected compact hypersurface of M, the relative volume of (P, M) is the quotient volume$(P)/$ volume(M). In this paper we give a comparison theorem for the relative volume of such a pair, with some bounds on the Ricci curvature of M and the mean curvature of P, with respect to that of a **model pair** $(\mathcal{P}, \mathcal{M})$ where $\mathcal{M}$ is a revolution manifold and $\mathcal{P}$ a "parallel" of $\mathcal{M}$.

1. Introduction

The "relative volume" is defined in the following way: Let $\mathcal{B}$ be a set of pairs (P, M), where M is an n-dimensional connected compact Riemannian manifold and P is a connected compact submanifold of M. The **relative volume** is the function $v: \mathcal{B} \longrightarrow \mathbb{R}^+$ defined by $v((P, M)) =$ volume$(P)/$ volume(M). Heintze and Karcher ([HK]) have got the infimum of such a function on sets $\mathcal{B}$ defined by

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constant bounds on the curvature of M and the mean curvature of P. Getting this infimum is equivalent to obtain a comparison theorem with the pair (P, M) on which the infimum is attained. This special pair is called the **model space**. In the result cited above and others more recent ([GM], [Gi], [MP]), the manifold M in the model space had constant sectional or holomorphic sectional curvature. In this paper we get the infimum of v in a situation where M is not a space form (either real or complex), but a revolution manifold.

This kind of model has been previously considered in other related situations. Thus [El], [Ab] use revolution manifolds to get generalized Toponogov's theorems, [Gl] uses them to obtain an isoperimetric inequality which is sharp in the limit, and [Ga] to get some estimates of the first Dirichlet eigenvalue on geodesic spheres.

Before stating our result we have to describe the revolution manifolds that we shall take as a model. We get from [BG] and [Gl] the following definition:

1.1 Definition: A C^∞ compact Riemannian manifold $(M^\varphi; \langle\, ,\rangle)$ of dimension n is called a **compact revolution manifold** if there are two points $\mathcal{N}$ and $\mathcal{S}$ of M^φ, a real number $L > 0$, a function $\varphi\colon [0, 2L] \longrightarrow [0, +\infty[$ and a diffeomorphism $f\colon]0, 2L[\times S^{n-1} \longrightarrow M - \{\mathcal{N}, \mathcal{S}\}$ such that at each $(s, x) \in]0, 2L[\times S^{n-1}$ we have $f^*\langle\, ,\rangle = ds^2 + \varphi^2(s)\langle\, ,\rangle_{S^{n-1}}$.

$\mathcal{N}$ and $\mathcal{S}$ will be called the **North** and **South poles** of M^φ. ∎

1.2 Remark: The facts that M^φ and $\langle\, ,\rangle$ be C^∞ imply that φ must satisfy the following properties:

$$\varphi(0) = \varphi(2L) = 0, \qquad \varphi'(0) = 1, \qquad \varphi'(2L) = -1,$$

and all the derivatives of φ of even order at 0 and $2L$ must be zero (see [Be, page 96]). ∎

Definition 1.1 is equivalent to the following one (see [Be] and [BG] or [Gl]):

1.3 Definition: A compact revolution manifold M of dimension n is an embedded compact hypersurface of $\mathbb{R}^{n+1}$, endowed with the induced metric, and constructed in the following way: let $c\colon [0, 4L] \longrightarrow \mathbb{R}^{n+1}$ be a closed plane curve, parametrized by its arc length, without self-intersections and symmetric with respect to some axis Z contained in the plane where the curve lies, with $c(0) = c(4L)$, $c(2L) \in Z$; then M is the set of all points in $\mathbb{R}^{n+1}$ which are in the spheres of centre in

Z, contained in a hyperplane orthogonal to Z and passing through a point in $c([0, 2L])$.

We shall call c the **generating curve of** M. ∎

The function $\varphi(s)$ in 1.1 is related to the curve $c(s)$ in 1.3 by

$$\varphi(s) = \text{distance}(c(s), Z)$$

1.4 Definition: We shall say that a compact revolution manifold M^φ is **convex** if the function φ of Definition 1.1 is convex ($\varphi''(t) \leq 0$) or, equivalently, the curve c in Definition 1.3 is convex. ∎

Given a revolution manifold M^φ, for any geodesic γ starting from $\mathcal{N}$, parametrized by arc length, and any t, the sectional curvature at $\gamma(t)$ corresponding to any plane containing $\gamma'(t)$ does not depend on γ or on the plane, and it will be denoted by $\rho(t)$.

1.5 Definition: We shall say that a compact revolution manifold M^φ is symmetric if the curvature $\rho(t)$ is **symmetric** with respect to L (i.e. $\rho(2L - t) = \rho(t)$ or, equivalently, $\rho(L - t) = \rho(L + t)$). When M^φ is symmetric, we shall say that M^φ is **lengthened** if ρ is decreasing on the interval $[0, L]$, and M^φ will be called **flattened** if ρ is increasing on $[0, L]$. ∎

Let us observe that if M^φ is lengthened (resp. flattened), then $\rho'(0) = 0 = \rho'(L)$, and ρ has a maximum (resp. a minimum) at 0 and a minimum (resp. maximum) at L.

The graphic idea is that a compact lengthened convex revolution manifold looks like a revolution ellipsoid in which the axis of the generating ellipse is longer in the direction of the revolution axis, and a compact flattened convex revolution manifold like a revolution ellipsoid in which the axis of the generating ellipse is shorter in the direction of the revolution axis.

Let us observe that the functions φ and ρ can be extended to C^∞-functions on $\mathbb{R}$ by the expressions

$$(1.1) \qquad \varphi(t + 2nL) = (-1)^{[n/2]+1}\, \varphi(t) \qquad \text{and} \qquad \rho(t + 2nL) = \rho(t),$$

where $[n/2]$ denotes the integer part of $n/2$. Moreover ρ and φ are related by the formula

$$(1.2) \qquad \rho(t) = -\frac{\varphi''(t)}{\varphi(t)}.$$

From (1.2) and the theorem of existence and uniqueness of solutions of differential equations it follows that M^φ is symmetric if and only if $\varphi(2L-t)=\varphi(t)$ or, equivalently, $\varphi(L-t)=\varphi(L+t)$.

We shall denote by $B_R{}^\varphi$ (respectively $S_R{}^\varphi$) the geodesic ball (respectively sphere) in M^φ with radius R and center $\mathcal{N}$ or $\mathcal{S}$.

From now on, M will be an n-dimensional oriented connected compact Riemannian manifold and P a connected compact oriented hypersurface. N will denote the unit normal vector field along P defining the orientation, and H the associated mean curvature. By $c(N(p))$ we shall denote the distance from $p \in P$ to its nearest cut-point of P, that is, $c(N(p)) = \sup\{t>0; \text{distance}(P,\gamma_p(t))=t\}$, $\gamma_p(t)$ being the geodesic satisfying $\gamma_p(0)=p$ and $\gamma_p'(0)=N(p)$.

Now, we state our main result.

1.6 THEOREM: *Let M^φ be a compact lengthened convex revolution manifold with sectional curvature $\rho(t)$. Let $R\in[0,L]$, and let M and P be as before.*

If $\mathrm{Ric}(\gamma_p'(t),\gamma_p'(t)) \geq (n-1)\rho(R-t)$ for every t such that $-c(-N(p)) \leq t \leq c(N(p))$, and $|H| \leq \varphi'(R)/\varphi(R)$, then

$$v(P,M) \geq v(S_R^\varphi, M^\varphi), \tag{1.6.1}$$

Moreover, the equality implies that there is an isometry between M and M^φ sending P onto S_R^φ.

We do not know if there is a similar theorem when M^φ is a compact flattened convex revolution manifold. In this case our proof of Lemma 3.5 does not work because inequalities (b) and (c) in Lemma 3.3 occur in the opposite sense.

Roughly speaking, if we consider the family of models consisting of symmetric compact convex revolution manifolds with the same poles, we have the round sphere at the center, for which Heintze–Karcher's Theorem holds. We have extended this theorem for the models inside the sphere (lengthened revolution manifolds), remaining unsolved, to our knowledge, the case with models outside the sphere (flattened revolution manifolds).

Here a comment on the difference between the cases P a hypersurface and P of higher codimension is in order. In the former case the function with which we compare the volume element has a different behaviour in the two opposite directions normal to P, and, to get a global result, one direction has to dominate the other, whereas in the second case that function has the same behaviour in all the normal directions (see [HK]). This is the reason why Theorem 1.3 in [GM] is not valid without an additional assumption on the sign of certain normal curvatures. This kind of assumption is not necessary in Heintze–Karcher's Theorem. Then, in this paper we have shown a different and more complicated situation in which the comparison still holds in full generality. For a much better understanding of the problem it will be interesting to know if there is a comparison theorem taking flattened revolution manifolds as models.

2. Some notation and background

From now on M, P, γ_p, N, R and L will be as in Theorem 1.6.

For the curvature and the Riemann Christoffel tensor we shall adopt the following convention sign:

$$\mathbf{R}(X,Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X,Y]}Z \quad \text{and} \quad \mathbf{R}_{X,Y,Z,W} = \langle \mathbf{R}(X,Y)Z, W\rangle.$$

We shall denote by $\mathcal{N}P$ the (unit) normal bundle of P in M. L_N will denote the Weingarten map of P associated to N.

Given any fibre bundle B on P and $p \in P$, B_p will denote the fibre of B over $p \in P$.

Given any Riemannian vector bundle V on any manifold Q, we shall denote by $V(t)$ the set $\{\zeta \in V; |\zeta| = t\}$.

We shall use $'$ to denote indistinctly the ordinary and the covariant derivative. Its exact meaning will be clear from the context.

First we recall some necessary background. Given $p \in P$ and the geodesic $\gamma_p(t)$ of M, the Jacobi operator $\mathcal{A}(t)$ is defined as the map

$$\mathcal{A}(t)\colon T_pP \longrightarrow T_pP \quad \text{such that} \quad \mathcal{A}(t)e = \tau_t^{-1}Y(t),$$

where $Y(t)$ is the P-Jacobi field along $\gamma_N(t)$ such that $Y(0) = e$ and $\nabla_N Y + L_N e = 0$, and τ_t is the parallel transport along $\gamma_p(t)$.

If $f(N(p)) = \inf\{t > 0; \gamma_p(t) \text{ is a focal point of } P\}$, then

$$c(N(p)) \le f(N(p)) = \inf\{t > 0; \operatorname{rank} \mathcal{A}(t) < n-1\}, \tag{2.2}$$

and

$$\operatorname{rank} \mathcal{A}(t) = \operatorname{rank} \exp_{\mathcal{N}P*tN} - 1 = \operatorname{rank} \exp_{\mathcal{N}P(t)*tN}, \tag{2.3}$$

where $\exp_{\mathcal{B}}$ denotes the restriction of the exponential map to the subset $\mathcal{B}$ of TM.

$S(t)$ will denote the Weingarten map of the tubular hypersurface of radius t about P with respect to the unit normal vector $\gamma_p'(t)$. The operators $\mathcal{A}(t)$ and $S(t)$ are related by

$$S(t) = -\mathcal{A}'(t)\mathcal{A}^{-1}(t) \tag{2.4}$$

as follows from [Ka, (1.2.6)] and is explicitly written in [CV] for geodesic spheres. Moreover $S(t)$ satisfies the equation (cfr. [Gr, page 36])

$$S'(t) = S^2(t) + \mathcal{R}(t), \tag{2.5}$$

where

$$\mathcal{R}(t)X = \mathbf{R}(\gamma_p'(t), X)\gamma_p'(t), \quad \text{for every} \quad X \in \{\gamma_p'(t)\}^{\perp}.$$

Let $\theta(p,t)$ be the real function defined on $\{(p,t) \in P \times \mathbb{R}; -c(-N(p)) < t < c(N(p))\}$ by $\omega(\gamma_p(t)) = \theta(p,t)dp \wedge dt$, ω being the riemannian volume element of M and dp that of P. Then θ satisfies the differential equation (see [Gr, page 42])

$$\frac{\theta'(p,t)}{\theta(p,t)} = -\operatorname{tr} S(t). \tag{2.6}$$

3. Some preliminary lemmas

In this paragraph, φ and ρ will be as in Theorem 1.6, that is, M^{φ} will be a compact lengthened convex revolution manifold.

3.1 Lemma: *Let $\zeta \in [0, L]$. The Riccati equation*

$$f'(t) = f^2(t) - \frac{\varphi''(\zeta - t)}{\varphi(\zeta - t)} \tag{3.1.1}$$

has the following general solution:

$$f(t) = -\frac{a_\zeta'(t) - f(0)b_\zeta'(t)}{a_\zeta(t) - f(0)b_\zeta(t)}, \tag{3.1.2}$$

where

(3.1.3)
$$a_\zeta(t) = \frac{\varphi(\zeta - t)}{\varphi(\zeta)} + \frac{\varphi'(\zeta)}{\varphi(\zeta)}b_\zeta(t) \quad \text{and} \quad b_\zeta(t) = \varphi(\zeta - t)\varphi(\zeta)\int_0^t \frac{1}{\varphi^2(\zeta - s)}ds.$$

Proof: An obvious particular solution of (3.1.1) is

$$f(t) = \frac{\varphi'(\zeta - t)}{\varphi(\zeta - t)}.$$

Hence, from the standard methods to solve Riccati equations (see [Eg] for instance), it follows that the general solution has the form

$$f(t) = \frac{\varphi'(\zeta - t)}{\varphi(\zeta - t)} + \frac{\frac{\varphi^2(\zeta)}{\varphi^2(\zeta - t)}}{\frac{1}{f(0) - \frac{\varphi'(\zeta)}{\varphi(\zeta)}} - \int_0^t \frac{\varphi^2(\zeta)}{\varphi^2(\zeta - s)} ds}$$

and (3.1.2) follows by a straightforward computation. ∎

3.2 Remark: The integrand $1/\varphi^2(\zeta - s)$ has a pole at $s = \zeta$, and the integral $\int_0^t \frac{1}{\varphi^2(\zeta - s)} ds$ is infinite for $t \geq \zeta$. Then, in the preceding lemma, this integral has to be understood as the difference between the values in t and 0 of the primitive function of $1/\varphi^2(\zeta - s)$. With this convention, it follows from the expression (3.1.3) that $b_\zeta(t)$ is well defined and C^∞ for $t \neq \zeta + n\,2L$, and having account of the following integration by parts

$$\begin{aligned}\int_0^t \frac{1}{\varphi^2(\zeta - s)} ds &= \int_0^t \frac{\varphi'(\zeta - s)}{\varphi^2(\zeta - s)} \frac{1}{\varphi'(\zeta - s)} ds \\ &= \frac{1}{\varphi(\zeta - s)} \frac{1}{\varphi'(\zeta - s)} \Big|_0^t - \int_0^t (\frac{1}{\varphi} \frac{\varphi''}{\varphi'^2})(\zeta - s) ds \\ &= \frac{1}{\varphi(\zeta - t)} \frac{1}{\varphi'(\zeta - t)} - \frac{1}{\varphi(\zeta)} \frac{1}{\varphi'(\zeta)} + \int_0^t \frac{1}{\varphi'^2(\zeta - s)} \rho(\zeta - s) ds\end{aligned}$$

we have that $b_\zeta(t)$ is well defined and C^∞ for $t \neq (2n+1)L + \zeta$. Then $a_\zeta(t)$ and $b_\zeta(t)$ are C^∞ and well defined everywhere.

Notation: Given a function q: $X \times \mathbb{R} \longrightarrow \mathbb{R}$, where X is a given space, we denote by $z^+(q)$ the function which associates, to every $x \in X$, the first zero of the function $t \longrightarrow q(x,t)$ for $t \in \mathbb{R}^+$, and by $z^-(q)$ the function which associates, to every $x \in X$, minus the first zero of the function $t \longrightarrow q(x, -t)$ for $t \in \mathbb{R}^+$. ∎

3.3 LEMMA: *The functions $a_\zeta(t)$ and $b_\zeta(t)$ defined in 3.1 (see also 3.2), and the functions $\alpha_\zeta(t) = a_\zeta(-t)$ and $\beta_\zeta(t) = -b_\zeta(-t)$ satisfy:*

(a) $a_\zeta(0) = 1$, $b_\zeta(0) = 0$, $a'_\zeta(0) = 0$, $b'_\zeta(0) = 1$;

(b) $z^+(a_\zeta) \leq z^+(\alpha_\zeta)$ *and* $a_\zeta(t) \leq \alpha_\zeta(t)$ *for every* $t \in [0, z^+(a_\zeta)]$, *and* $\zeta \in [0, L]$;

$z^+(a_\zeta) \geq z^+(\alpha_\zeta)$ *and* $a_\zeta(t) \geq \alpha_\zeta(t)$ *for every* $t \in [0, z^+(\alpha_\zeta)]$, *and* $\zeta \in [L, 2L]$;

(c) $z^+(b_\zeta) \leq z^+(\beta_\zeta)$ *and* $b_\zeta(t) \leq \beta_\zeta(t)$ *for every* $t \in [0, z^+(b_\zeta)]$, *and* $\zeta \in [0, L]$;
$z^+(b_\zeta) \geq z^+(\beta_\zeta)$ *and* $b_\zeta(t) \geq \beta_\zeta(t)$ *for every* $t \in [0, z^+(\beta_\zeta)]$, *and* $\zeta \in [L, 2L]$;

(d) $z^+(a_\zeta) \leq z^+(b_\zeta)$ *and* $z^+(\alpha_\zeta) \leq z^+(\beta_\zeta)$.

Proof: (a) follows from (3.1.3) by a direct computation. In order to prove (b), let us observe that from Lemma 3.1 it follows that $\nu_\zeta(t) = a_\zeta(t) - h\, b_\zeta(t)$ satisfies the differential equation

$$\nu_\zeta''(t) + \rho(\zeta - t)\,\nu_\zeta(t) = 0, \tag{3.3.1}$$

for any h such that $|h| \leq |\frac{\varphi'}{\varphi}(\zeta)|$. Then, taking $h = 0$, we see that $a_\zeta(t)$ satisfies the differential equation (3.3.1) with the initial conditions given by (a).

On the other hand, because $\alpha_\zeta'(t) = -a_\zeta'(-t)$ and $\alpha_\zeta''(t) = a_\zeta''(-t)$, α satisfies the differential equation

$$\alpha_\zeta''(t) + \rho(\zeta + t)\,\alpha_\zeta(t) = 0 \quad \text{with} \quad \alpha_\zeta(0) = 1, \quad \alpha_\zeta'(0) = 0. \tag{3.3.2}$$

Since M^φ is lengthened (i.e. $\rho(2L - t) = \rho(t)$ and ρ is decreasing in $[0, L]$), using the formula (1.1), we have that

(3.3.3)
$$\text{if} \quad t \in \bigcup\{[2nL, (2n+1)L],\, n \in \mathbb{Z}\}, \begin{cases} \rho(\zeta + t) \leq \rho(\zeta - t) & \text{for} \quad \zeta \in [0, L], \\ \rho(\zeta + t) \geq \rho(\zeta - t) & \text{for} \quad \zeta \in [L, 2L], \end{cases}$$

and

$$\alpha_\zeta''(t) = -\rho(\zeta + t)\alpha_\zeta(t) < 0, \quad a_\zeta''(t) = -\rho(\zeta - t)a_\zeta(t) < 0 \tag{3.3.4}$$
$$\text{for} \quad t \in [0, \inf\{z^+(a_\zeta), z^+(\alpha_\zeta)\}].$$

All this allows us to prove the assertion (b) by an argument similar to the proof of Sturm's comparison theorem (see [DC, pages 238 ff.] or [Kl, pages 150–151]) if we can show the inequality

$$\int_0^t (\rho(\zeta - s) - \rho(\zeta + s))\; a_\zeta(s)\,\alpha_\zeta(s)\,ds \begin{cases} \geq 0, & \text{if } \zeta \in [0, L], \\ \leq 0, & \text{if } \zeta \in [L, 2L], \end{cases} \tag{3.3.5}$$

which is the main fact in the Sturm's argument that is not obvious in our context. To show it, we shall use the facts that ρ is periodic of period $2L$ (see(1.1)), that a_ζ and α_ζ are decreasing in $[0, \inf\{z^+(a_\zeta), z^+(\alpha_\zeta)\}]$, as follows from (3.3.4), and the inequalities (3.3.3). Indeed, using these facts we can do the following computation: let $n = [(t-L)/L]$ (where $[c]$ means the integer part of c), if $n = 2m$, $m \in \mathbb{Z}$, then

$$\begin{aligned}
&\int_0^t (\rho(\zeta - t) - \rho(\zeta + t))\, a_\zeta(t)\, \alpha_\zeta(t) dt \\
&= \sum_{i=0}^{m-1} \Big(\int_{2iL}^{(2i+1)L} (\rho(\zeta - t) - \rho(\zeta + t))\, a_\zeta(t)\, \alpha_\zeta(t) dt \\
&\qquad + \int_{(2i+1)L}^{(2i+2)L} (\rho(\zeta - t) - \rho(\zeta + t))\, a_\zeta(t)\, \alpha_\zeta(t) dt \Big) \\
&\qquad + \int_{2mL}^{t} (\rho(\zeta - t) - \rho(\zeta + t))\, a_\zeta(t)\, \alpha_\zeta(t) dt \\
&= \sum_{i=0}^{m-1} \Big(\int_0^L (\rho(\zeta - ((2i+1)L - s)) - \rho(\zeta + ((2i+1)L - s))) \\
&\qquad\qquad \cdot a_\zeta((2i+1)L - s)\, \alpha_\zeta((2i+1)L - s) ds \\
&\qquad + \int_0^L (\rho(\zeta - ((2i+1)L + s)) - \rho(\zeta + ((2i+1)L + s))) \\
&\qquad\qquad \cdot a_\zeta((2i+1)L + s)\, \alpha_\zeta((2i+1)L + s) ds \Big) \\
&\qquad + \int_0^{t-2mL} (\rho(\zeta - s) - \rho(\zeta + s))\, a_\zeta(s + 2mL)\, \alpha_\zeta(s + 2mL) ds \\
&= \sum_{i=0}^{m-1} \int_0^L (\rho(\zeta - ((2i+1)L - s)) - \rho(\zeta + (2i+1)L - s)) \\
&\qquad \cdot \big(a_\zeta((2i+1)L - s)\, \alpha_\zeta((2i+1)L - s) \\
&\qquad\qquad - a_\zeta((2i+1)L + s)\, \alpha_\zeta((2i+1)L + s) \big) ds \\
&\qquad + \int_0^{t-2mL} (\rho(\zeta - s) - \rho(\zeta + s))\, a_\zeta(s + 2mL)\, \alpha_\zeta(s + 2mL) ds \\
&\qquad\qquad \begin{cases} \geq 0 & \text{if } \zeta \in [0, L] \\ \leq 0 & \text{if } \zeta \in [L, 2L]; \end{cases}
\end{aligned}$$

if $n = 2m + 1$, $m \in \mathbb{Z}$, then

$$\begin{aligned}
&\int_0^t (\rho(\zeta - t) - \rho(\zeta + t))\, a_\zeta(t)\, \alpha_\zeta(t) dt \\
&= \sum_{i=0}^{m-1} \Big(\int_{2iL}^{(2i+1)L} (\rho(\zeta - t) - \rho(\zeta + t))\, a_\zeta(t)\, \alpha_\zeta(t) dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{(2i+1)L}^{(2i+2)L} (\rho(\zeta-t)-\rho(\zeta+t))\, a_\zeta(t)\,\alpha_\zeta(t)\,dt\ )\\
&+\int_{2mL}^{(2m+1)L-(t-(2m+1)L)} (\rho(\zeta-t)-\rho(\zeta+t))\, a_\zeta(t)\,\alpha_\zeta(t)\,dt\\
&+\int_{(2m+1)L-(t-(2m+1)L)}^{(2m+1)L} (\rho(\zeta-t)-\rho(\zeta+t))\, a_\zeta(t)\,\alpha_\zeta(t)\,dt\\
&+\int_{(2m+1)L}^{t} (\rho(\zeta-t)-\rho(\zeta+t))\, a_\zeta(t)\,\alpha_\zeta(t)\,dt\\
=&\sum_{i=0}^{m-1}\int_0^L (\rho(\zeta-((2i+1)L-s))-\rho(\zeta+((2i+1)L-s)))\\
&\qquad\cdot(\ a_\zeta((2i+1)L-s)\,\alpha_\zeta((2i+1)L-s)\\
&\qquad\qquad -a_\zeta((2i+1)L+s)\,\alpha_\zeta((2i+1)L+s)\ )\ ds\\
&+\int_{2mL}^{(2m+1)L-(t-(2m+1)L)} (\rho(\zeta-t)-\rho(\zeta+t))\, a_\zeta(t)\,\alpha_\zeta(t)\,dt\\
&+\int_0^{t-(2m+1)L} (\rho(\zeta-((2m+1)L-s))-\rho(\zeta+(2m+1)L-s))\\
&\qquad\cdot(\ a_\zeta(2mL+L-s)\,\alpha_\zeta(2mL+L-s)\\
&\qquad\qquad -a_\zeta(2mL+L+s)\,\alpha_\zeta(2mL+L+s)\ )\ ds\\
&\qquad\qquad\begin{cases}\geq 0 & \text{if}\quad \zeta\in[0,L],\\ \leq 0 & \text{if}\quad \zeta\in[L,2L].\end{cases}
\end{aligned}
$$

This finishes the proof of (3.3.5), and hence, the proof of (b). To prove (c) , let us remark that if we take the derivative with respect to h in (3.3.1), then we have that $b_\zeta(t)$ satisfies the same differential equation (3.3.1) with the initial conditions given by (a), and, moreover, because $\beta'_\zeta(t)=b'_\zeta(-t)$, and $\beta''_\zeta(t)=-b''_\zeta(-t)$, we have that β_ζ satisfies the differential equation (3.3.2) with the initial conditions $\beta_\zeta(0)=0$, and $\beta'_\zeta(0)=1$. From these remarks, the proof of (c) follows from similar arguments to the proof of (b). The assertion (d) is a consequence of the Sturm's oscillation Theorem (see [DC, pages 240–241] or [CL, page 208]). ∎

3.4 LEMMA: *Let M, P, R, and $\gamma_p(t)$ be as in Theorem 1.6. If* $\mathrm{Ric}(\gamma'_p(t),\gamma'_p(t))\geq (n-1)\rho(R-t)$ *for every t such that* $-c(-N(p))\leq t\leq c(N(p))$, *then*

$$\theta(p,t)\leq \mu_R(H(p),t), \tag{3.4.1}$$

where

$$\mu_\zeta(h,t)=(a_\zeta(t)-h\,b_\zeta(t))^{n-1}. \tag{3.4.2}$$

The equality in (3.4.1) is attained if and only if P is umbilical with mean curvature $\varphi'(R)/\varphi(R)$ and the sectional curvature of M for the planes containing $\gamma_p'(t)$ at the point $\gamma_p(t)$ is $\rho(R-t)$.

Proof: Let $\{E_i(t)\}_{i=1}^{n-1}$ be a parallel orthonormal frame of $\{\gamma_p'(t)\}^\perp$ along $\gamma_p(t)$. Let us consider the functions

$$f_i(t) = \langle S(t)E_i(t), E_i(t)\rangle.$$

Taking the derivative of these functions and using the equation (2.5) for $S(t)$ and the Cauchy–Schwarz inequality, we get

(3.4.3)
$$\begin{aligned} f_i' = \langle S'E_i, E_i\rangle &= \langle S^2E_i + \mathcal{R}(t)E_i, E_i\rangle \\ &= \|SE_i\|^2 + \langle \mathcal{R}(t)E_i, E_i\rangle \\ &\geq \langle SE_i, E_i\rangle^2 + \langle \mathcal{R}(t)E_i, E_i\rangle = f_i^2 + \langle \mathcal{R}(t)E_i, E_i\rangle. \end{aligned}$$

But, from the hypothesis on the Ricci curvature, we have

$$\sum_{i=1}^{n-1}\langle \mathcal{R}(t)E_i, E_i\rangle \geq (n-1)\,\rho(R-t) = (n-1)\,\rho(2L-R+t).$$

Then we have the differential inequalities

$$\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i\right)' \geq \frac{1}{n-1}\sum_{i=1}^{n-1} f_i^2 + \rho(R-t) \geq \left(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i\right)^2 + \rho(R-t);$$

with

$$\frac{1}{n-1}\sum_{i=1}^{n-1} f_i(0) = H(p).$$

Then, from [EH] or [Ro, Theorem 5] and Lemma 3.1, we have

$$\frac{1}{n-1}\sum_{i=1}^{n-1} f_i(t) \geq -\frac{a_R'(t) - H(p)\,b_R'(t)}{a_R(t) - H(p)\,b_R(t)}.$$

Then, from this inequality and (2.6) we get

$$\frac{\theta'(p,t)}{\theta(p,t)} = -\operatorname{tr} S(t) \leq \frac{d}{dt}\ln\left(a_R(t) - H(p)\,b_R(t)\right)^{n-1}.$$

Integration between 0 and t gives

$$\ln\frac{\theta(p,t)}{\theta(p,0)} \le \ln\left(\frac{a_R(t)-H(p)\,b_R(t)}{a_R(0)-H(p)\,b_R(0)}\right)^{n-1};$$

whence (3.4.1) follows because $\theta(p,0)=1=a(0)$ and $b(0)=0$. The equality in (3.4.1) implies the equality in (3.4.3), which implies that $E_i(t)$ are eigenvectors of $S(t)$, with eigenvalue the function $f(t)$ in Lemma 3.1. Then the Riccati equation (2.5) for S implies the claimed expression for $\mathcal{R}(t)$. ∎

3.5 LEMMA: *The function* $f\colon [-\frac{\varphi'(R)}{\varphi(R)}, \frac{\varphi'(R)}{\varphi(R)}] \longrightarrow \mathbb{R}$ *defined by*

$$f(h)=\int_{z^-(\mu)}^{z^+(\mu)} \mu(h,t)dt, \quad \text{with } \mu(h,t)=\begin{cases} \mu_R(h,t) \text{ if } h\in\left[0,\dfrac{\varphi'(R)}{\varphi(R)}\right] \\ \mu_{2L-R}(h,t) \text{ if } h\in\left[-\dfrac{\varphi'(R)}{\varphi(R)},0\right], \end{cases}$$

is increasing on the interval $[0,\frac{\varphi'(R)}{\varphi(R)}]$ *and decreasing on the interval* $[-\frac{\varphi'(R)}{\varphi(R)},0]$.

Proof: In the following we always consider $\mu_R(h,t)$ defined only for $h\in[0,\frac{\varphi'(R)}{\varphi(R)}]$ and $\mu_{2L-R}(h,t)$ defined only for $h\in[-\frac{\varphi'(R)}{\varphi(R)},0]$. Computing the derivative of f respect to h gives

$$\frac{df}{dh}=\int_{z^-(\mu)}^{z^+(\mu)}\frac{\partial\mu(h,t)}{\partial h}dt,$$

with

$$\frac{\partial\mu_\zeta(h,t)}{\partial h}=-(n-1)\,(a_\zeta(t)-h\,b_\zeta(t))^{n-2}\,b_\zeta(t).$$

Then, since $a_\zeta(t)-h\,b_\zeta(t)=\mu_\zeta(h,t)^{1/(n-1)}$, $\mu_\zeta(h,0)=1=a_\zeta(0)$, $b_\zeta(0)=0$ and $b'_\zeta(0)=1$, we have that

$$\frac{\partial\mu(h,t)}{\partial h}\le 0 \quad \text{if } t\in[0,z^+(\mu)] \quad\text{and}\quad \frac{\partial\mu(h,t)}{\partial h}\ge 0 \quad \text{if } t\in[z^-(\mu),0].$$

We shall prove that $df/dh\ge 0$ on $[0,\frac{\varphi'(R)}{\varphi(R)}]$ and $df/dh\le 0$ on $[-\frac{\varphi'(R)}{\varphi(R)},0]$ showing that

$$\text{(3.5.1)}\qquad z^+(\mu_R)\le -z^-(\mu_R), \qquad -z^-(\mu_{2L-R})\le z^+(\mu_{2L-R})$$

and

$$\text{(3.5.2)}\qquad \int_{-z^+(\mu_R)}^{z^+(\mu_R)}\frac{\partial\mu_R(h,t)}{\partial h}dt\ge 0, \qquad \int_{z^-(\mu_{2L-R})}^{-z^-(\mu_{2L-R})}\frac{\partial\mu_{2L-R}(h,t)}{\partial h}dt\le 0.$$

In order to prove (3.5.1), let us observe that $z^+(\mu) = z^+(\mu(h,t)^{1/(n-1)})$ and $z^-(\mu) = -z^+(\mu(h,-t)^{1/(n-1)})$; then it is sufficient to prove that

$$\begin{aligned} z^+(\mu_R(h,t)^{1/(n-1)}) &\le z^+(\mu_R(h,-t)^{1/(n-1)}) \\ z^+(\mu_{2L-R}(h,t)^{1/(n-1)}) &\ge z^+(\mu_{2L-R}(h,-t)^{1/(n-1)}) \\ \text{and} \qquad \mu(h,t)^{1/(n-1)} &\le \mu(h,-t)^{1/(n-1)}. \end{aligned}$$

But

$$\begin{aligned} \mu_R(h,t)^{1/(n-1)} &\le \mu_R(0,t)^{1/(n-1)}, \\ \mu_R(0,-t)^{1/(n-1)} &\le \mu_R(h,-t)^{1/(n-1)}; \\ \mu_{2L-R}(h,t)^{1/(n-1)} &\ge \mu_{2L-R}(0,t)^{1/(n-1)}, \quad \text{and} \\ \mu_{2L-R}(0,-t)^{1/(n-1)} &\ge \mu_{2L-R}(h,-t)^{1/(n-1)}. \end{aligned}$$

Then it suffices to prove that

$$\begin{aligned} a_R(t) = \mu_R(0,t)^{1/(n-1)} &\le \mu_R(0,-t)^{1/(n-1)} \\ &= a_R(-t) = \alpha_R(t) \quad \text{for } t \in [0, z^+(a_R)], \end{aligned}$$

and

$$\begin{aligned} a_{2L-R}(t) = \mu_{2L-R}(0,t)^{1/(n-1)} &\ge \mu_{2L-R}(0,-t)^{1/(n-1)} \\ &= a_{2L-R}(-t) = \alpha_{2L-R}(t) \quad \text{for } t \in [0, z^+(\alpha_{2L-R})], \end{aligned}$$

but this is just given by Lemma 3.3 (b).

To prove (3.5.2), let us observe that

$$\int_{-z^+(\mu)}^{z^+(\mu)} \frac{\partial \mu(h,t)}{\partial h} dt = \int_0^{z^+(\mu)} \frac{\partial(\mu(h,t) + \mu(h,-t))}{\partial h} dt.$$

and

$$\mu_\zeta(h,-t) = (a_\zeta(-t) - h\, b_\zeta(-t))^{n-1} = (\alpha_\zeta(t) + h\,\beta_\zeta(t))^{n-1}.$$

Then

$$\begin{aligned} \mu_\zeta(h,-t) + \mu_\zeta(h,t) =& (\alpha_\zeta + h\,\beta_\zeta)^{n-1} + (a_\zeta - h\, b_\zeta)^{n-1} \\ =& \sum_{i=0}^{n-1} \binom{n-1}{i} \alpha_\zeta^i\, \beta_\zeta^{n-1-i}\, h^{n-1-i} \\ &+ \sum_{i=0}^{n-1} \binom{n-1}{i} a_\zeta^i\, b_\zeta^{n-1-i}\, h^{n-1-i} (-1)^{n-1-i} \\ =& \sum_{i=0}^{n-1} \binom{n-1}{i} (\alpha_\zeta^i\, \beta_\zeta^{n-1-i} + (-1)^{n-1-i} a_\zeta^i\, b_\zeta^{n-1-i})\, h^{n-1-i}. \end{aligned}$$

Now, by Lemma 3.3 (b), (c) and (d), we have that $\alpha_R^i \beta_R^{n-1-i} \geq a_R^i b_R^{n-1-i}$ and $\alpha_{2L-R}^i \beta_{2L-R}^{n-1-i} \leq a_{2L-R}^i b_{2L-R}^{n-1-i}$, then

$$\frac{\partial\left(\mu_R(h,t)+\mu_R(h,-t)\right)}{\partial h} \geq 0 \quad \text{and} \quad \frac{\partial\left(\mu_{2L-R}(h,t)+\mu_{2L-R}(h,-t)\right)}{\partial h} \leq 0,$$

which finishes the proof, because if $h \leq 0$, then $h^{n-1-i-1} \leq 0$ when $n-1-i$ is even. ∎

4. Proof of Theorem 1.6

First, let us observe that from Lemma 3.4 it follows that when $P = S_R^\varphi$ and $M = M^\varphi$, then $\theta(p,t) = \mu(\pm\frac{\varphi'(R)}{\varphi(R)}, t)$, where we must take the sign $+$ if S_R^φ is centred at $\mathcal{N}$ with normal vector pointing inward, and the sign $-$ if S_R^φ is centred at $\mathcal{S}$ with normal vector pointing outward (i.e. S_R^φ is considered as S_{2L-R}^φ centred at $\mathcal{N}$ with normal vector pointing inward). On the other hand it is easy to check the identity

$$\mu_R(H(p),t) = \mu_{2L-R}(H(p),-t). \tag{4.1}$$

Let $P_+ = \{p \in P,\ H(p) \geq 0\}$ and $P_- = \{p \in P,\ H(p) < 0\}$; then, from (2.2), the definition of $\theta(p,t)$, lemmas 3.4 and 3.5, formula (4.1) and the above observation, we have

$$\begin{aligned}
\mathrm{vol}(M) &= \int_P \int_{-c(-N(p))}^{c(N(p))} \theta(p,t)\,dt\,dp \leq \int_P \int_{z^-(\theta(p,t))}^{z^+(\theta(p,t))} \theta(p,t)\,dt\,dp \\
&= \int_{P_+} \int_{z^-(\theta(p,t))}^{z^+(\theta(p,t))} \theta(p,t)\,dt\,dp + \int_{P_-} \int_{z^-(\theta(p,t))}^{z^+(\theta(p,t))} \theta(p,t)\,dt\,dp \\
&\leq \int_{P_+} \int_{z^-(\mu(H(p),t))}^{z^+(\mu(H(p),t))} \mu(H(p),t)\,dt\,dp + \\
&\quad \int_{P_-} \int_{z^-(\mu(H(p),-t))}^{z^+(\mu(H(p),-t))} \mu(H(p),-t)\,dt\,dp \\
&\leq \int_{P_+} \int_{z^-(\mu(\frac{\varphi'(R)}{\varphi(R)},t))}^{z^+(\mu(\frac{\varphi'(R)}{\varphi(R)},t))} \mu(\frac{\varphi'(R)}{\varphi(R)},t)\,dt\,dp \\
&\quad + \int_{P_-} \int_{z^-(\mu(-\frac{\varphi'(R)}{\varphi(R)},-t))}^{z^+(\mu(-\frac{\varphi'(R)}{\varphi(R)},-t))} \mu(-\frac{\varphi'(R)}{\varphi(R)},-t)\,dt\,dp \\
&= \int_{P_+} \int_{z^-(\mu(\frac{\varphi'(R)}{\varphi(R)},t))}^{z^+(\mu(\frac{\varphi'(R)}{\varphi(R)},t))} \mu(\frac{\varphi'(R)}{\varphi(R)},t)\,dt\,dp
\end{aligned}$$

$$+\int_{P_-}\int_{z^-(\mu(-\frac{\varphi'(R)}{\varphi(R)},t))}^{z^+(\mu(-\frac{\varphi'(R)}{\varphi(R)},t))}\mu(-\frac{\varphi'(R)}{\varphi(R)},t)\,dt\,dp$$
$$=\int_{P_+}\frac{1}{\mathrm{vol}(S_R^\varphi)}\mathrm{vol}(M^\varphi)dp+\int_{P_-}\frac{1}{\mathrm{vol}(S_R^\varphi)}\mathrm{vol}(M^\varphi)dp$$
$$=\frac{\mathrm{vol}(P)}{\mathrm{vol}(S_R^\varphi)}\mathrm{vol}(M^\varphi),$$

which is the inequality (1.6.1).

Equality implies $H(p)=\frac{\varphi'(R)}{\varphi(R)}$ or $H(p)=-\frac{\varphi'(R)}{\varphi(R)}$ for every $p\in P$.

If $H(p)=\frac{\varphi'(R)}{\varphi(R)}$, then equality also implies

$$c(N(p))=f(N(p))=R,$$
$$c(-N(p))=f(-N(p))=2L-R,$$
$$\theta(p,t)=\mu\left(\frac{\varphi'(R)}{\varphi(R)},t\right)$$

and, from Lemma 3.2, $\mathcal{R}(t)=\rho(R-t)\,\mathrm{Id}$. Then, from (2.4) and (2.5), we get $\mathcal{A}(t)=\varphi(R-t)\,\mathrm{Id}$. Then, from (2.3), $\mathrm{rank}(\exp_{\mathcal{N}P(R)*RN(p)})=0$ for every $p\in P$. This implies that there is a point $m\in M$ such that

$$\exp_{\mathcal{N}P}(\{RN(p),\,p\in P\})=\{m\}.$$

Let us define the map

$$F\colon P\longrightarrow S^{n-1}\subset T_mM\qquad\text{such that } F(p)=-\gamma_p'(R).$$

Its derivative has the following form. Let c be a curve in P such that $c(0)=p$ and $c'(0)=X$, then

$$F_{*p}(X)=-\frac{d}{ds}\gamma'_{c(s)}(R)|_{s=0}=-\frac{\nabla}{ds}\frac{\partial}{\partial t}\exp_{c(s)}t\,N(c(s))|_{s=0,t=R}$$
$$=-\tau_R\mathcal{A}'(R)X=-\tau_R(-\varphi'(0)X)=\tau_R(X).$$

Then F is a local diffeomorfism, and $F(P)$ is open in S^{n-1}. On the other hand, since P is compact, $F(P)$ is closed in S^{n-1}. Then $F(P)=S^{n-1}$, i.e. for every $\xi\in S^{n-1}$ there is a $p\in P$ such that $\xi=\gamma_p'(R)$. Then the geodesic sphere with center m and radius R of M,

$$S_m(R)=\exp_m(\{R\xi/\xi\in S^{n-1}\})=\{\exp_p((R-R)N(p),p\in P\}=P.$$

Since the P-Jacobi fields along γ_p vanish at m, they are also $\{m\}$-Jacobi fields along $\gamma_\xi(t) = \gamma_p(R-t)$. Then the Jacobi operators on M for the geodesics starting from m are the same that those in M^φ. Then, from the expression of the metric tensor in geodesic spherical coordinates (cfr [Ch, page 317]), M is isometric to M^φ and, obviously, the isometry takes P onto S_R^φ.

If $H(p) = -\frac{\varphi'(R)}{\varphi(R)}$, the equality implies

$$c(N(p)) = f(N(p)) = 2L - R, \quad c(-N(p)) = f(-N(p)) = R,$$

$$\theta(p,t) = \mu\left(-\frac{\varphi'(R)}{\varphi(R)}, t\right),$$

$$\mathcal{R}(t) = \rho(2L - R + t)\,\mathrm{Id},\ \mathcal{R}(-t) = \rho(2L - R + t)\,\mathrm{Id} = \rho(R-t)\,\mathrm{Id},$$

and the rest of the above argument works only changing t by $-t$ and R by $-R$.

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